

The range of a linear map

1. The span of a collection of vectors is a subspace of the vector space in which they are defined. A basis for that span can be found by creating the corresponding matrix, performing Gaussian elimination until a matrix that is row-echelon form is found, and then choosing those vectors that correspond to columns with leading non-zero entries.

Given the following five vectors, what is a basis for the span given the information provided?

$$\begin{pmatrix} -0.8 \\ 0.2 \\ 2.0 \\ 0.6 \end{pmatrix}, \begin{pmatrix} -1.2 \\ 0.3 \\ 3.0 \\ 0.9 \end{pmatrix}, \begin{pmatrix} -1.4 \\ 3.2 \\ 2.0 \\ 0.9 \end{pmatrix}, \begin{pmatrix} 2.2 \\ 1.4 \\ 4.0 \\ -0.7 \end{pmatrix}, \begin{pmatrix} 4.2 \\ 2.1 \\ 1.0 \\ -2.0 \end{pmatrix} \text{ and } \begin{pmatrix} -0.8 & -1.2 & -1.4 & 2.2 & 4.2 \\ 0.2 & 0.3 & 3.2 & 1.4 & 2.1 \\ 2.0 & 3.0 & 2.0 & 4.0 & 1.0 \\ 0.6 & 0.9 & 0.9 & -0.7 & -2.0 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & 2 & 4 & 1 \\ 0 & 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

2. What is a basis for the range and the dimension of the range for the matrix $A: \mathbf{R}^5 \rightarrow \mathbf{R}^4$ where

$$A = \begin{pmatrix} -0.8 & -1.2 & -1.4 & 2.2 & 4.2 \\ 0.2 & 0.3 & 3.2 & 1.4 & 2.1 \\ 2.0 & 3.0 & 2.0 & 4.0 & 1.0 \\ 0.6 & 0.9 & 0.9 & -0.7 & -2.0 \end{pmatrix}?$$

3. Is the span of the five vectors in Question 1 all of $\mathbf{R}^4$?

4. Is the range of the linear map A in Question 2 all of the co-domain? That is, is A onto?

5. For each of the following functions that map $\mathbf{R}$ to $\mathbf{R}$, choose those that are onto:

$$e^x, x^2, x^3, x^3 + x, x^3 - x, \sin(x)$$

6. If $B: \mathbf{C}^4 \rightarrow \mathbf{C}^3$, answer the following:

- Suppose solving $B\mathbf{u} = \mathbf{0}_3$ has one free variable, is B onto and what is the dimension of the range?
- Suppose solving $B\mathbf{u} = \mathbf{0}_3$ has two free variables, is B onto and what is the dimension of the range?
- Suppose solving $B\mathbf{u} = \mathbf{0}_3$ has four free variables, what matrix is B and what is the dimension of the range?
- Can B have no free variables and if so, what is the dimension of the range?